REPRESENTATION THEORY OF FRAMISATIONS OF KNOT ALGEBRAS

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Abstract. We study the algebraic structure and the representation theory of the Yokonuma–Hecke algebra of type A, its generalisations, the affine and cyclotomic Yokonuma–Hecke algebras, and its Temperley–Lieb type quotients, the Yokonuma–Temperley–Lieb algebra, the Framisation of the Temperley–Lieb algebra and the Complex Reflection Temperley–Lieb algebra.

1. Introduction

A knot algebra is an algebra obtained as a quotient of the group algebra of a braid group and endowed with a Markov trace. Using Jones's method [Jo2, Jo3] of normalising and re-scaling the trace according to the braid equivalence, these algebras can be used for the definition of knot invariants. Some known examples are the Temperley–Lieb algebra (Jones polynomial), the Iwahori–Hecke algebra of type A (Homflypt polynomial), the Iwahori–Hecke algebra of type B (Geck–Lambropoulou invariants), the Ariki–Koike algebras and the affine Hecke algebra of type A (Lambropoulou invariants), the singular Hecke algebra (Kauffman–Vogel & Paris–Rabenda invariants), the BMW algebra (Kauffman polynomial) and the Rook algebra (Alexander polynomial).

Modular framisation (or simply framisation) is a mechanism proposed recently by Juyumaya and Lambropoulou [JuLa5] which consists of constructing a non-trivial extension of a knot algebra via the addition of the so-called "framing" generators, each of which is a generator of a cyclic group. In this way we obtain a new algebra which is related to framed braids and framed knots.

The inspiring example of framisation is the Yokonuma–Hecke algebra of type A. Yokonuma–Hecke algebras were introduced by Yokonuma [Yo] in the context of finite reductive groups as generalisations of Iwahori–Hecke algebras. Given a finite reductive group G, the Iwahori–Hecke algebra is the endomorphism ring of the permutation representation of G with respect to a Borel subgroup, while the Yokonuma–Hecke algebra is the endomorphism ring of the permutation representation of G with respect to a maximal unipotent subgroup.

In recent years, the presentation of the Yokonuma–Hecke algebra of type A has been transformed in [Ju1, JuKa, Ju2, ChPdA1, ChPo2] to the one that we will use here. This new presentation is given by generators and relations, depending on two positive integers, d and n, and a parameter q. For $q = p^m$ and $d = p^m - 1$, where p is a prime number and m is a positive integer, the Yokonuma–Hecke algebra of type A, denoted by $\mathrm{Y}_{d,n}(q)$, is the endomorphism ring of the permutation representation of $\mathrm{GL}_n(\mathbb{F}_q)$ with respect to a maximal unipotent subgroup. The algebra $\mathrm{Y}_{d,n}(q)$ can be viewed as a framisation of the Iwahori–Hecke algebra $\mathcal{H}_n(q) \cong \mathrm{Y}_{1,n}(q)$, whose presentation we deform by adding the framing generators $t_1, \ldots, t_n$, which generate the finite abelian group $(\mathbb{Z}/d\mathbb{Z})^n$. Thus, $\mathrm{Y}_{d,n}(q)$ can be obtained as a quotient of the group algebra of both the framed braid group $\mathbb{Z}^n \rtimes B_n$ and the modular framed braid group $(\mathbb{Z}/d\mathbb{Z})^n \rtimes B_n$, where B_n denotes the classical braid group on n strands. Finally, $\mathrm{Y}_{d,n}(q)$ can be also obtained as a deformation of the group algebra of the complex reflection group $G(d,1,n) \cong (\mathbb{Z}/d\mathbb{Z})^n \rtimes \mathfrak{S}_n$, different from the Ariki–Koike algebra [ArKo].

Juyumaya [Ju2] has defined a Markov trace on $\mathrm{Y}_{d,n}(q)$, which has been subsequently used by himself and Lambropoulou for the definition of 2-variable isotopy invariants for framed [JuLa1, JuLa2], classical [JuLa3] and singular [JuLa4] knots and links, after Jones's method. The invariants for classical links are

The author would like to thank the organisers of the conference "Algebraic modeling of topological and computational structures and applications", and in particular Professor Sofia Lambropoulou, who has inspired this research. A grateful acknowledgement to the referee whose suggestions led us to improve this paper, as well as [ChPo2]. This research has been co-financed by the European Union (European Social Fund - ESF) and Greek national funds through the Operational Program "Education and Lifelong Learning" of the National Strategic Reference Framework (NSRF) - Research Funding Program: THALIS. .

simply the invariants for framed links restricted to links with all framings equal to 0 (that is, the links obtained as closures of elements of B_n). Using the new presentation for $\mathrm{Y}_{d,n}(q)$ established in [ChPdA1], we have recently proved that the classical link invariants obtained from the Yokonuma–Hecke algebra are not topologically equivalent to the Homflypt polynomial [CJKL]. This implies that framisations of knot algebras are very useful to topologists not only for the construction of framed link invariants, but also for the construction of new classical link invariants.

We have now introduced and studied many interesting new algebras, which are obtained as framisations of other important knot algebras. First of all, we have the affine and cyclotomic Yokonuma–Hecke algebras [ChPdA2], which generalise respectively the affine Hecke algebras of type A and the Ariki–Koike algebras. In fact, we have shown in [ChSe] that affine Yokonuma–Hecke algebras appear also naturally in the study of p-adic reductive groups, arising from a construction analogous to the one used by Yokonuma, while cyclotomic Yokonuma–Hecke algebras give rise to both Ariki–Koike algebras and classical Yokonuma–Hecke algebras of type A, both as quotients and as particular cases. Further, we have three possible framisations of the Temperley–Lieb algebra, all obtained as quotients of $\mathrm{Y}_{d,n}(q)$ by a suitable two-sided ideal: the Yokonuma–Temperley–Lieb algebra [GJKL1], the Framisation of the Temperley–Lieb algebra [GJKL2] and the Complex Reflection Temperley–Lieb algebra [GJKL2]. Our study of the structure and the representation theory of these three algebras in [ChPo1, ChPo2] indicates that the Framisation of the Temperley–Lieb algebra is the most natural analogue of the Temperley–Lieb algebra in this setting.

Now, all algebras mentioned above are endowed with Markov traces (see [ChPdA2] for the affine and cyclotomic Yokonuma–Hecke algebras, and [GJKL1, GJKL2] for the Temperley–Lieb quotients of the Yokonuma–Hecke algebra), which can be used for the definition of knot invariants after Jones's method. In view of the results of [CJKL], we have concluded that the invariants for links in the solid torus obtained from the affine and cyclotomic Yokonuma–Hecke algebras in [ChPdA2] are not topologically equivalent to the invariants obtained from the affine and cyclotomic Hecke algebras in [La1, GeLa, La2], whereas the 1-variable invariants obtained from the Framisation of the Temperley–Lieb algebra in [GJKL2] are not topologically equivalent to the Jones polynomial.

In view of the importance of the algebras mentioned above to both algebraists and topologists, in this paper we will study their algebra structure and representation theory. We will provide explicit combinatorial formulas for their irreducible representations, compute their dimensions, construct bases for them and give semisimplicity criteria. We will also discuss their symmetric algebra structure.

2. Symmetric algebras

Let R be a commutative integral domain and let A be an R-algebra, free and finitely generated as an R-module. If R' is a commutative integral domain containing R, we will write $R'A$ for $R' \otimes_R A$ and we will denote by $\mathrm{Irr}(R'A)$ the set of irreducible representations of $R'A$.

A *symmetrising form* on the algebra A is a linear map $\tau : A \to R$ such that

(a) $\tau(ab) = \tau(ba)$ for all $a, b \in A$, that is, τ is a trace function, and

(b) the map $\widehat{\tau} : A \to \mathrm{Hom}_R(A, R)$, $a \mapsto (x \mapsto \tau(ax))$ is an isomorphism of A-bimodules.

If there exists a symmetrising form on A, we say that A is a *symmetric* algebra.

Example 2.1. Let G be a finite group. The linear map $\tau : \mathbb{Z}[G] \to \mathbb{Z}$ defined by $\tau(1) = 1$ and $\tau(g) = 0$ for all $g \in G \setminus \{1\}$ is a symmetrising form on $\mathbb{Z}[G]$; it is called the *canonical symmetrising form* on $\mathbb{Z}[G]$.

Suppose that there exists a symmetrising form τ on A. Let K be a field containing R such that the algebra KA is split. The map τ can be extended to KA by extension of scalars. If $V \in \mathrm{Irr}(KA)$ and χ_V denotes the corresponding irreducible character, then $\widehat{\tau}^{-1}(\chi_V)$ belongs to the centre of KA [GePf, Lemma 7.1.7]. Schur's lemma implies that $\widehat{\tau}^{-1}(\chi_V)$ acts as a scalar on V; we define this scalar to be the *Schur element* associated with V and denote it by s_V. We have $s_V \in R_K$, where R_K denotes the integral closure of R in K [GePf, Proposition 7.3.9].

Example 2.2. Let G be a finite group and let τ be the canonical symmetrising form on $A := \mathbb{Z}[G]$. If K is an algebraically closed field of characteristic 0, then KA is a split semisimple algebra and $s_V = |G|/\chi_V(1)$ for all $V \in \mathrm{Irr}(KA)$.

Following [GePf, Theorem 7.2.6], we have that the algebra KA is semisimple if and only if $s_V \neq 0$ for all $V \in \mathrm{Irr}(KA)$. If this is the case,

$$\tau = \sum_{V \in \mathrm{Irr}(KA)} \frac{1}{s_V} \chi_V.$$

From now on, we assume that R is integrally closed in K. Let $\theta : R \to L$ be a ring homomorphism into a field L such that L is the field of fractions of $\theta(R)$. We call such a ring homomorphism a *specialisation* of R. Schur elements can be then used to determine whether the algebra LA is semisimple as follows [GePf, Theorem 7.4.7]:

Theorem 2.3. *Assume that KA and LA are split and that A is symmetric with symmetrising form τ. For any simple KA-module V, let $s_V \in R$ be the Schur element with respect to τ. Then LA is semisimple if and only if $\theta(s_V) \neq 0$ for all $V \in \mathrm{Irr}(KA)$.*

Finally, if LA is semisimple, we have the following famous result known as "Tits's deformation theorem". For its proof, the reader may refer, for example, to [GePf, Theorem 7.4.6].

Theorem 2.4. *Assume that KA and LA are split. If LA is semisimple, then KA is also semisimple and we have a bijection $\mathrm{Irr}(KA) \leftrightarrow \mathrm{Irr}(LA)$.*

3. Yokonuma–Hecke algebras

Let $n \in \mathbb{N}$, $d \in \mathbb{N}^*$. Let q be an indeterminate. The *Yokonuma–Hecke algebra (of type A)*, denoted by $\mathrm{Y}_{d,n}(q)$, is an associative $\mathbb{C}[q, q^{-1}]$-algebra generated by the elements

$$g_1, \ldots, g_{n-1}, t_1, \ldots, t_n$$

subject to the following relations:

$$(3.1)\qquad \begin{array}{llcll} (\mathrm{b}_1) & g_i g_j & = & g_j g_i & \text{for all } i,j = 1,\ldots,n-1 \text{ such that } |i-j|>1,\\ (\mathrm{b}_2) & g_i g_{i+1} g_i & = & g_{i+1} g_i g_{i+1} & \text{for all } i = 1,\ldots,n-2,\\ (\mathrm{f}_1) & t_i t_j & = & t_j t_i & \text{for all } i,j = 1,\ldots,n,\\ (\mathrm{f}_2) & t_j g_i & = & g_i t_{s_i(j)} & \text{for all } i = 1,\ldots,n-1 \text{ and } j = 1,\ldots,n,\\ (\mathrm{f}_3) & t_j^d & = & 1 & \text{for all } j = 1,\ldots,n, \end{array}$$

where s_i is the transposition $(i, i+1)$, together with the quadratic relations:

$$(3.2)\qquad g_i^2 = q + (q-1)\, e_i\, g_i \qquad \text{for all } i = 1,\ldots,n-1,$$

where

$$(3.3)\qquad e_i := \frac{1}{d} \sum_{s=0}^{d-1} t_i^s t_{i+1}^{-s}.$$

It is easily verified that the elements e_i are idempotents in $\mathrm{Y}_{d,n}(q)$. Also, that the elements g_i are invertible, with

$$(3.4)\qquad g_i^{-1} = q^{-1} g_i + (q^{-1} - 1)\, e_i \qquad \text{for all } i = 1,\ldots,n-1.$$

If we specialise q to 1, the defining relations (3.1)–(3.2) become the defining relations for the complex reflection group $G(d,1,n) \cong (\mathbb{Z}/d\mathbb{Z}) \wr \mathfrak{S}_n \cong (\mathbb{Z}/d\mathbb{Z})^n \rtimes \mathfrak{S}_n$. Thus, the algebra $\mathrm{Y}_{d,n}(q)$ is a deformation of the group algebra over $\mathbb{C}$ of $G(d,1,n)$. Moreover, for $d = 1$, the Yokonuma–Hecke algebra $\mathrm{Y}_{1,n}(q)$ coincides with the Iwahori–Hecke algebra $\mathcal{H}_n(q)$ of type A, and thus, for $d = 1$ and q specialised to 1, we obtain the group algebra over $\mathbb{C}$ of the symmetric group $\mathfrak{S}_n$.

Remark 3.1. Note that in all the papers prior to [ChPdA1], the algebra $\mathrm{Y}_{d,n}(q)$ is generated by elements $\overline{g}_1, \ldots, \overline{g}_{n-1}, t_1, \ldots, t_n$ satisfying relations (3.1) and the quadratic relations:

$$(3.5)\qquad \overline{g}_i^2 = 1 + (q-1)\, e_i + (q-1)\, e_i\, \overline{g}_i \qquad \text{for all } i = 1,\ldots,n-1.$$

This presentation changed in [ChPdA1], where we considered $\mathrm{Y}_{d,n}(q)$ defined over $\mathbb{C}[q^{1/2}, q^{-1/2}]$ and generated by elements $\widetilde{g}_1, \ldots, \widetilde{g}_{n-1}, t_1, \ldots, t_n$ satisfying relations (3.1) and the quadratic relations:

$$(3.6)\qquad \widetilde{g}_i^2 = 1 + (q^{1/2} - q^{-1/2})\, e_i\, \widetilde{g}_i \qquad \text{for all } i = 1,\ldots,n-1.$$

By taking $\overline{g}_i := \widetilde{g}_i + (q^{1/2} - 1)\, e_i \widetilde{g}_i$ (and thus, $\widetilde{g}_i = \overline{g}_i + (q^{-1/2} - 1)\, e_i\, \overline{g}_i$), we obtain the old presentation of the Yokonuma–Hecke algebra. By taking $g_i := q^{1/2} \widetilde{g}_i$ we obtain our presentation of the Yokonuma–Hecke algebra.

Now let $w \in \mathfrak{S}_n$, and let $w = s_{i_1} s_{i_2} \dots s_{i_r}$ be a reduced expression for w. Since the generators g_i of the Yokonuma–Hecke algebra satisfy the same braid relations as the generators of $\mathfrak{S}_n$, Matsumoto's theorem (see, for example, [GePf, Theorem 1.2.2]) implies that the element $g_w := g_{i_1} g_{i_2} \dots g_{i_r}$ is well defined, that is, it does not depend on the choice of the reduced expression of w.

Juyumaya [Ju2] has proved that the following set is a $\mathbb{C}[q, q^{-1}]$-basis of $\mathrm{Y}_{d,n}(q)$:

$$\mathcal{B}^{\mathrm{H}}_{d,n} := \{\, t_1^{r_1} \dots t_n^{r_n} g_w \mid w \in \mathfrak{S}_n,\ 0 \leqslant r_j \leqslant d-1 \text{ for all } j = 1, 2, \dots, n \}\,. \tag{3.7}$$

In particular, $\mathrm{Y}_{d,n}(q)$ is a free $\mathbb{C}[q, q^{-1}]$-module of rank $d^n n!$.

The representation theory of Yokonuma–Hecke algebras has been first studied by Thiem [Th1, Th2, Th3] in the general context of unipotent Hecke algebras. The generality of his results and the new presentation for $\mathrm{Y}_{d,n}(q)$ has led us to develop in [ChPdA1] a combinatorial approach to the representation theory of the Yokonuma–Hecke algebra of type A, in terms of d-partitions and standard d-tableaux.

3.1. **Combinatorics of d-partitions and standard d-tableaux.** A *partition* $\lambda = (\lambda_1, \dots, \lambda_h)$ is a family of positive integers such that $\lambda_1 \geqslant \lambda_2 \geqslant \dots \geqslant \lambda_h \geqslant 1$. We write $|\lambda| := \sum_{i=1}^{h} \lambda_i$ and we say that λ is a partition of n if $n = |\lambda|$. We denote by $\mathcal{P}(n)$ the set of partitions of n. We define the set of nodes $[\lambda]$ of λ to be the set

$$[\lambda] := \{(x, y) \mid 1 \leqslant x \leqslant h,\ 1 \leqslant y \leqslant \lambda_x\}.$$

We identify partitions with their Young diagrams: the Young diagram of λ is a left-justified array of h rows such that the i-th row contains λ_i boxes (nodes) for all $i = 1, \dots, h$.

A *d-partition* of n is an ordered d-tuple $\boldsymbol{\lambda} = (\lambda^{(0)}, \lambda^{(1)}, \dots, \lambda^{(d-1)})$ of partitions such that $|\boldsymbol{\lambda}| := \sum_{i=0}^{d-1} |\lambda^{(i)}| = n$. We denote by $\mathcal{P}(d, n)$ the set of d-partitions of n. The empty multipartition, denoted by $\boldsymbol{\emptyset}$, is a d-tuple of empty partitions. A node $\boldsymbol{\theta}$ of $\boldsymbol{\lambda}$ is a triple (x, y, i), where $0 \leqslant i \leqslant d-1$ and (x, y) is a node of the partition $\lambda^{(i)}$. We define $\mathrm{p}(\boldsymbol{\theta}) := i$ to be the *position* of $\boldsymbol{\theta}$ and $\mathrm{c}(\boldsymbol{\theta}) := q^{y-x}$ to be the *quantum content* of $\boldsymbol{\theta}$.

A *d-tableau of shape $\boldsymbol{\lambda}$* is a bijection between the set $\{1, \dots, n\}$ and the set of nodes of $\boldsymbol{\lambda}$. In other words, a d-tableau of shape $\boldsymbol{\lambda}$ is obtained by placing the numbers $1, \dots, n$ in the nodes of $\boldsymbol{\lambda}$. The *size* of a d-tableau of shape $\boldsymbol{\lambda}$ is n, that is, the size of $\boldsymbol{\lambda}$. A d-tableau is *standard* if its entries increase along any row and down any column of every diagram in $\boldsymbol{\lambda}$. For $d = 1$, a standard 1-tableau is a usual standard tableau.

For a d-tableau $\mathcal{T}$, we denote respectively by $\mathrm{p}(\mathcal{T}|i)$ and $\mathrm{c}(\mathcal{T}|i)$ the position and the quantum content of the node with the number i in it. For example, for the standard 3-tableau $\mathcal{T} = \big(\boxed{1}\boxed{3}\,, \varnothing\,, \boxed{2}\big)$ of size 3, we have

$$\mathrm{p}(\mathcal{T}|1) = 0\,,\ \ \mathrm{p}(\mathcal{T}|2) = 2\,,\ \ \mathrm{p}(\mathcal{T}|3) = 0 \qquad \text{and} \qquad \mathrm{c}(\mathcal{T}|1) = 1\,,\ \ \mathrm{c}(\mathcal{T}|2) = 1\,,\ \ \mathrm{c}(\mathcal{T}|3) = q\,.$$

For any d-tableau $\mathcal{T}$ of size n and any permutation $\sigma \in \mathfrak{S}_n$, we denote by $\mathcal{T}^\sigma$ the d-tableau obtained from $\mathcal{T}$ by applying the permutation σ on the numbers contained in the nodes of $\mathcal{T}$. We have

$$\mathrm{p}(\mathcal{T}^\sigma|i) = \mathrm{p}\big(\mathcal{T}|\sigma^{-1}(i)\big) \quad \text{and} \quad \mathrm{c}(\mathcal{T}^\sigma|i) = \mathrm{c}\big(\mathcal{T}|\sigma^{-1}(i)\big) \qquad \text{for all } i = 1, \dots, n.$$

Note that if the d-tableau $\mathcal{T}$ is standard, the d-tableau $\mathcal{T}^\sigma$ is not necessarily standard.

3.2. **Representation theory of Yokonuma–Hecke algebras.** Let $\boldsymbol{\lambda} \in \mathcal{P}(d, n)$, and let $V_{\boldsymbol{\lambda}}$ be a $\mathbb{C}(q)$-vector space with a basis $\{\mathbf{v}_{\mathcal{T}}\}$ indexed by the standard d-tableaux of shape $\boldsymbol{\lambda}$. We set $\mathbf{v}_{\mathcal{T}} := 0$ for any non-standard d-tableau $\mathcal{T}$ of shape $\boldsymbol{\lambda}$. By [ChPdA1, Proposition 5 & Theorem 1] and [ChPo2, Theorem 3.7], we have the following description of the irreducible representations of $\mathbb{C}(q)\mathrm{Y}_{d,n}(q)$:

Theorem 3.2. *Let $\{\xi_0, \xi_1, \dots, \xi_{d-1}\}$ be the set of all d-th roots of unity (ordered arbitrarily). Let $\mathcal{T}$ be a standard d-tableau of shape $\boldsymbol{\lambda} \in \mathcal{P}(d, n)$. For brevity, we set $\mathrm{p}_i := \mathrm{p}(\mathcal{T}|i)$ and $\mathrm{c}_i := \mathrm{c}(\mathcal{T}|i)$ for all $i = 1, \dots, n$. The vector space $V_{\boldsymbol{\lambda}}$ is a representation of $\mathbb{C}(q)\mathrm{Y}_{d,n}(q)$ with the action of the generators on the basis element $\mathbf{v}_{\mathcal{T}}$ defined as follows: for $j = 1, \dots, n$,*

$$t_j(\mathbf{v}_{\mathcal{T}}) = \xi_{\mathrm{p}_j} \mathbf{v}_{\mathcal{T}}\ ; \tag{3.8}$$

for $i = 1, \dots, n-1$, if $\mathrm{p}_i > \mathrm{p}_{i+1}$ then

$$g_i(\mathbf{v}_{\mathcal{T}}) = \mathbf{v}_{\mathcal{T}^{s_i}}\ , \tag{3.9}$$

if $\mathrm{p}_i < \mathrm{p}_{i+1}$ *then*

$$g_i(\mathbf{v}_\mathcal{T}) = q\,\mathbf{v}_{\mathcal{T}^{s_i}}\ , \tag{3.10}$$

and if $\mathrm{p}_i = \mathrm{p}_{i+1}$ *then*

$$g_i(\mathbf{v}_\mathcal{T}) = \frac{q\mathrm{c}_{i+1} - \mathrm{c}_{i+1}}{\mathrm{c}_{i+1} - \mathrm{c}_i}\,\mathbf{v}_\mathcal{T} + \frac{q\mathrm{c}_{i+1} - \mathrm{c}_i}{\mathrm{c}_{i+1} - \mathrm{c}_i}\,\mathbf{v}_{\mathcal{T}^{s_i}}\ , \tag{3.11}$$

where s_i *is the transposition* $(i, i+1)$. *Further, the set* $\{V_{\boldsymbol{\lambda}} \,|\, \boldsymbol{\lambda} \in \mathcal{P}(d,n)\}$ *is a complete set of pairwise non-isomorphic irreducible representations of* $\mathbb{C}(q)\mathrm{Y}_{d,n}(q)$.

The above theorem implies that the algebra $\mathbb{C}(q)\mathrm{Y}_{d,n}(q)$ is split. As we have already mentioned, when $q \mapsto 1$, the algebra $\mathbb{C}(q)\mathrm{Y}_{d,n}(q)$ specialises to the group algebra $\mathbb{C}[G(d,1,n)]$, which is semisimple. By Tits's deformation theorem, we obtain that the algebra $\mathbb{C}(q)\mathrm{Y}_{d,n}(q)$ is also semisimple.

Let now $\theta : \mathbb{C}[q, q^{-1}] \to \mathbb{C}$ be a ring homomorphism such that $\theta(q) = \eta \in \mathbb{C} \setminus \{0\}$. Using the representation theory of $\mathbb{C}(q)\mathrm{Y}_{d,n}(q)$, we have proved the following semisimplicity criterion for $\mathbb{C}\mathrm{Y}_{d,n}(\eta)$ [ChPdA1, Proposition 9]:

Proposition 3.3. *The specialised Yokonuma–Hecke algebra* $\mathbb{C}\mathrm{Y}_{d,n}(\eta)$ *is (split) semisimple if and only if* $\theta(P(q)) \neq 0$, *where*

$$P(q) = \prod_{1 \leqslant i \leqslant n} (1 + q + \cdots + q^{i-1}).$$

Note that following Ariki's semisimplicity criterion [Ar] for Ariki–Koike algebras (and so, in particular, for Iwahori–Hecke algebras of type A), the algebra $\mathbb{C}\mathrm{Y}_{d,n}(\eta)$ is semisimple if and only if the specialised Iwahori–Hecke algebra $\mathbb{C}\mathcal{H}_n(\eta)$ is semisimple.

Another way to obtain the above result is through our definition of a canonical symmetrising form τ on $\mathrm{Y}_{d,n}(q)$ [ChPdA1, Proposition 10]. Having calculated the Schur elements of $\mathrm{Y}_{d,n}(q)$ with respect to τ [ChPdA1, Proposition 11], we can deduce the above semisimplicity criterion with the use of Theorem 2.3. More precisely, we have the following:

Theorem 3.4. *We define the linear map* $\tau : \mathrm{Y}_{d,n}(q) \to \mathbb{C}[q, q^{-1}]$ *by*

$$\tau(t_1^{r_1} \dots t_n^{r_n} g_w) = \begin{cases} 1 & \text{if } w = 1 \text{ and } r_j = 0 \text{ for all } j = 1, 2, \dots, n, \\ 0 & \text{otherwise,} \end{cases} \tag{3.12}$$

where $w \in \mathfrak{S}_n$ *and* $0 \leqslant r_j \leqslant d-1$ *for all* $j = 1, 2, \dots, n$. *Then* τ *is a symmetrising form on* $\mathrm{Y}_{d,n}(q)$, *called the canonical symmetrising form. If* $\boldsymbol{\lambda} = (\lambda^{(0)}, \dots, \lambda^{(d-1)}) \in \mathcal{P}(d,n)$, *then the Schur element of* $V_{\boldsymbol{\lambda}}$ *with respect to* τ *is*

$$s_{\boldsymbol{\lambda}} = d^n\, s_{\lambda^{(0)}} s_{\lambda^{(1)}} \dots s_{\lambda^{(d-1)}}\ , \tag{3.13}$$

where $s_{\lambda^{(i)}}$ *is the Schur element of the Iwahori–Hecke algebra* $\mathcal{H}_{|\lambda^{(i)}|}(q)$ *corresponding to* $\lambda^{(i)}$ *for all* $i = 0, 1, \dots, d-1$ *(we take* $s_\emptyset := 1$*).*

The Schur elements of Iwahori–Hecke algebras of type A have been calculated by Steinberg [St]. A simple formula for them is given by Jacon and the author in [ChJa].

The connection between the representation theory of the Yokonuma–Hecke algebra and that of Iwahori–Hecke algebras of type A implied by (3.13) is explained by a result of Lusztig [Lu, §34], who has proved that Yokonuma–Hecke algebras, in general, are isomorphic to direct sums of matrix algebras over certain subalgebras of classical Iwahori–Hecke algebras. Using the new presentation for $\mathrm{Y}_{d,n}(q)$, Jacon and Poulain d'Andecy [JaPdA] have explicitly described this isomorphism between the Yokonuma–Hecke algebra of type A and a direct sum of matrix algebras over tensor products of Iwahori–Hecke algebras of type A. Another proof of this result has been given recently in [EsRy], where Espinoza and Ryom-Hansen have constructed a concrete isomorphism between $\mathrm{Y}_{d,n}(q)$ and Shoji's modified Ariki–Koike algebra. Note that in all cases the result has been obtained over the ring $\mathbb{C}[q^{1/2}, q^{-1/2}]$ (using the generators $\widetilde{g}_i$ defined in Remark 3.1). We have managed to show that it is still valid over the smaller ring $\mathbb{C}[q, q^{-1}]$. We have [ChPo2, Theorem 4.3]:

$$\mathrm{Y}_{d,n}(q) \cong \bigoplus_{\mu \in \mathrm{Comp}_d(n)} \mathrm{Mat}_{m_\mu}(\mathcal{H}_{\mu_0}(q) \otimes \mathcal{H}_{\mu_1}(q) \otimes \cdots \otimes \mathcal{H}_{\mu_{d-1}}(q)), \tag{3.14}$$

where

$$\mathrm{Comp}_d(n) = \{\mu = (\mu_0, \mu_1, \dots, \mu_{d-1}) \in \mathbb{N}^d \mid \mu_0 + \mu_1 + \dots + \mu_{d-1} = n\} \tag{3.15}$$

and

$$m_\mu = \frac{n!}{\mu_0!\mu_1!\dots\mu_{d-1}!} \tag{3.16}$$

4. Affine and Cyclotomic Yokonuma–Hecke algebras

In [ChPdA2], we introduced the affine and cyclotomic Yokonuma–Hecke algebras, which give rise to both Ariki–Koike algebras and Yokonuma–Hecke algebras of type A as quotients and as special cases. Let $n \in \mathbb{N}$, $d \in \mathbb{N}^*$ and $l \in \mathbb{N}^* \cup \{\infty\}$. Let q and $(Q_i)_{i\in\mathbb{N}}$ be indeterminates, and set $\mathcal{R}_l := \mathbb{C}[q^{\pm1}, Q_0^{\pm1}, Q_1^{\pm1}, \dots, Q_{l-1}^{\pm1}]$ if $l < \infty$, and $\mathcal{R}_\infty := \mathbb{C}[q^{\pm1}]$. We define the algebra $\mathrm{Y}(d, l, n)$ to be the associative $\mathcal{R}_l$-algebra generated by the elements

$$g_1, \dots, g_{n-1}, t_1, \dots, t_n, X_1, X_1^{-1}$$

subject to the relations (3.1)–(3.2), together with the following relations concerning the generator X_1:

$$\begin{array}{rclll} X_1\, X_1^{-1} & = & X_1^{-1} X_1 & = & 1 \\ X_1\, g_1 X_1 g_1 & = & g_1 X_1 g_1\, X_1 & & \\ X_1 g_i & = & g_i X_1 & & \text{for all } i = 2, \dots, n-1, \\ X_1 t_j & = & t_j X_1 & & \text{for all } j = 1, \dots, n, \end{array} \tag{4.1}$$

and if $l < \infty$,

$$(X_1 - Q_0)(X_1 - Q_1) \cdots (X_1 - Q_{l-1}) = 0. \tag{4.2}$$

The algebra $\mathrm{Y}(d, \infty, n)$ is called the *affine Yokonuma–Hecke algebra*. For $l < \infty$, the algebra $\mathrm{Y}(d, l, n)$ is called the *cyclotomic Yokonuma–Hecke algebra*. These algebras are isomorphic to the modular framisations of, respectively, the affine Hecke algebra ($l = \infty$) and the Ariki–Koike algebra ($l < \infty$); see definitions in [JuLa5, Section 6] and [ChPdA1, Remark 1].

The cyclotomic Yokonuma–Hecke algebra is a quotient of the affine Yokonuma–Hecke algebra by the relation (4.2). If we map $X_1 \mapsto Q_0$ for $l < \infty$ or $X_1 \mapsto 1$ for $l = \infty$, we obtain a surjection of $\mathrm{Y}(d, l, n)$ onto $\mathrm{Y}_{d,n}(q)$. If we map $t_j \mapsto 1$ for all $j = 1, \dots, n$, then we obtain a surjection of $\mathrm{Y}(d, l, n)$ onto $\mathcal{H}(l, n)$, where $\mathcal{H}(l, n)$ denotes the Ariki–Koike algebra associated to $G(l, 1, n)$ for $l < \infty$ and $\mathcal{H}(\infty, n)$ denotes the affine Hecke algebra of type A. Moreover, we have $\mathrm{Y}(d, 1, n) \cong \mathrm{Y}_{d,n}(q)$ and $\mathrm{Y}(1, l, n) \cong \mathcal{H}(l, n)$. In particular, we have $\mathrm{Y}(1, 1, n) \cong \mathcal{H}_n(q)$.

Remark 4.1. Let p be a prime number. In a recent series of papers [Vi1, Vi2, Vi3], Vignéras introduced and studied a large family of algebras, called *pro-p-Iwahori–Hecke algebras*. They generalise convolution algebras of compactly supported functions on a p-adic connected reductive group that are bi-invariant under the pro-p-radical of an Iwahori subgroup, which play an important role in the p-modular representation theory of p-adic reductive groups. In [ChSe] we have shown that the affine Yokonuma–Hecke algebra $\mathrm{Y}(d, \infty, n)$ is a pro-p-Iwahori–Hecke algebra. Thus, the affine Yokonuma–Hecke algebra generalises the affine Hecke algebra of type A in a similar way that the Yokonuma–Hecke algebra generalises the Iwahori–Hecke algebra of type A. In particular, for $q = p^m$ and $d = p^m - 1$, where m is a positive integer, $\mathrm{Y}(d, \infty, n)$ is isomorphic to the convolution algebra of complex valued and compactly supported functions on the group $\mathrm{GL}_n(F)$, with F a suitable p-adic field, that are bi-invariant under the pro-p-radical of an Iwahori subgroup.

Remark 4.2. Following Lusztig's approach in [Lu], Cui [Cui] has established an explicit algebra isomorphism between the affine Yokonuma–Hecke algebra $\mathrm{Y}(d, \infty, n)$ and a direct sum of matrix algebras over tensor products of affine Hecke algebras of type A, similar to (3.14). More recently, Poulain d'Andecy [PdA] obtained the same result, as well as an isomorphism between the cyclotomic Yokonuma–Hecke algebra $\mathrm{Y}(d, l, n)$, where $l < \infty$, and a direct sum of matrix algebras over tensor products of Ariki–Koike algebras, using the same approach as in [JaPdA]. The isomorphism theorem for cyclotomic Yokonuma–Hecke algebras has been subsequently re-obtained by Rostam [Ro] using his result that cyclotomic Yokonuma–Hecke algebras are cyclotomic quiver Hecke algebras.

In [ChPdA2] we have constructed several bases for the algebra $\mathrm{Y}(d,l,n)$. In order to describe them here, we introduce the following notation: Let $Z_l := \{0,\dots,l-1\}$ for $l<\infty$ and $Z_\infty := \mathbb{Z}$. We define inductively elements $X_2,\dots,X_n$ of $\mathrm{Y}(d,l,n)$ by setting

$$X_{i+1} := q^{-1} g_i X_i g_i \quad \text{ for all } i=1,\dots,n-1.$$

Let $\mathcal{B}_{d,n}$ be a basis of the Yokonuma–Hecke algebra $\mathrm{Y}_{d,n}(q) \cong \mathrm{Y}(d,1,n)$ over $\mathcal{R}_l$ (we can take, for example, $\mathcal{B}^{\mathrm{H}}_{d,n}$ defined in (3.7)). We denote by $\mathcal{B}^{\mathrm{AK}}_{d,l,n}$ the following set of elements of $\mathrm{Y}(d,l,n)$:

$$X_1^{a_1}\dots X_n^{a_n}\cdot\omega\,, \quad a_k\in Z_l \text{ and } \omega\in\mathcal{B}_{d,n}.$$

Now, for $k=1,\dots,n$, we set

$$W^{(k)}_{J,a,b} := g_J^{-1}\dots g_2^{-1}g_1^{-1}X_1^a\,t_1^b\,g_1g_2\dots g_{k-1}\,,$$

$$W^{(k)-}_{J,a,b} := g_J\dots g_2g_1\,X_1^a\,t_1^b\,g_1^{-1}g_2^{-1}\dots g_{k-1}^{-1}\,,$$

$$\widetilde{W}^{(k)}_{J,a,b} := g_J\dots g_2g_1\,X_1^a\,t_1^b\,g_1g_2\dots g_{k-1}\,,$$

$$\widetilde{W}^{(k)-}_{J,a,b} := g_J^{-1}\dots g_2^{-1}g_1^{-1}\,X_1^a\,t_1^b\,g_1^{-1}g_2^{-1}\dots g_{k-1}^{-1}\,,$$

where $J\in\{0,\dots,k-1\}$ and $a,b\in\mathbb{Z}$. We use the following standard conventions: for $\epsilon=\pm1$, $g_J^\epsilon\dots g_2^\epsilon g_1^\epsilon := 1$ and $g_{k-J}^\epsilon\dots g_{k-2}^\epsilon g_{k-1}^\epsilon := 1$ if $J=0$. Then we denote, respectively, by $\mathcal{B}^{\mathrm{Ind}}_{d,l,n}$, $\mathcal{B}^{\mathrm{Ind}-}_{d,l,n}$, $\widetilde{\mathcal{B}}^{\mathrm{Ind}}_{d,l,n}$ and $\widetilde{\mathcal{B}}^{\mathrm{Ind}-}_{d,l,n}$ the following sets of elements of $\mathrm{Y}(d,l,n)$:

$$W^{(n)}_{J_n,a_n,b_n}\dots W^{(2)}_{J_2,a_2,b_2}W^{(1)}_{J_1,a_1,b_1}\,, \quad J_k\in\{0,\dots,k-1\},\ a_k\in Z_l \text{ and } b_k\in\{0,\dots,d-1\}.$$

$$W^{(n)-}_{J_n,a_n,b_n}\dots W^{(2)-}_{J_2,a_2,b_2}W^{(1)-}_{J_1,a_1,b_1}\,, \quad J_k\in\{0,\dots,k-1\},\ a_k\in Z_l \text{ and } b_k\in\{0,\dots,d-1\}.$$

$$\widetilde{W}^{(n)}_{J_n,a_n,b_n}\dots \widetilde{W}^{(2)}_{J_2,a_2,b_2}\widetilde{W}^{(1)}_{J_1,a_1,b_1}\,, \quad J_k\in\{0,\dots,k-1\},\ a_k\in Z_l \text{ and } b_k\in\{0,\dots,d-1\}.$$

$$\widetilde{W}^{(n)-}_{J_n,a_n,b_n}\dots \widetilde{W}^{(2)-}_{J_2,a_2,b_2}\widetilde{W}^{(1)-}_{J_1,a_1,b_1}\,, \quad J_k\in\{0,\dots,k-1\},\ a_k\in Z_l \text{ and } b_k\in\{0,\dots,d-1\}.$$

We then have the following [ChPdA2, Theorem 4.4]:

Theorem 4.3. *Each set $\mathcal{B}^{\mathrm{AK}}_{d,l,n}$, $\mathcal{B}^{\mathrm{Ind}}_{d,l,n}$, $\mathcal{B}^{\mathrm{Ind}-}_{d,l,n}$, $\widetilde{\mathcal{B}}^{\mathrm{Ind}}_{d,l,n}$ and $\widetilde{\mathcal{B}}^{\mathrm{Ind}-}_{d,l,n}$ is an $\mathcal{R}_l$-basis of $\mathrm{Y}(d,l,n)$. In particular, $\mathrm{Y}(d,l,n)$ is a free $\mathcal{R}_l$-module and, if $l<\infty$, its rank is equal to $(dl)^n n!$.*

Remark 4.4. The set $\mathcal{B}^{\mathrm{AK}}_{d,l,n}$ is the analogue of the Ariki–Koike basis of the Ariki–Koike algebra $\mathcal{H}(l,n)$ for $l<\infty$, and the standard Bernstein basis of the affine Hecke algebra of type A for $l=\infty$. The four other sets are inductive sets with respect to n, which are analogous to the inductive bases of $\mathcal{H}(l,n)$ studied in [La2, OgPo].

Furthermore, in [ChPdA2] we have studied the representation theory of the cyclotomic Yokonuma–Hecke algebra $\mathrm{Y}(d,l,n)$, which is quite similar to the representation theory of the Yokonuma–Hecke algebra $\mathrm{Y}_{d,n}(q)$. From now on, we only consider the case $l<\infty$.

Let $\mathcal{K}_l$ denote the field of fractions of $\mathcal{R}_l$. We will see that the irreducible representations of the algebra $\mathcal{K}_l\mathrm{Y}(d,l,n)$ are parametrised by the dl-partitions of n. Instead of looking though at dl-partitions as dl-tuples of partitions, we look at them as d-tuples of l-partitions, and we call them *(d,l)-partitions* when seen as such. We denote by $\mathcal{P}(d,l,n)$ the set of (d,l)-partitions of n. If $\boldsymbol{\lambda}\in\mathcal{P}(d,l,n)$, then $\boldsymbol{\lambda}=(\boldsymbol{\lambda}^{(0)},\boldsymbol{\lambda}^{(1)},\dots,\boldsymbol{\lambda}^{(d-1)})$, where $\boldsymbol{\lambda}^{(i)}$ is an l-partition for all $i=0,1,\dots,d-1$, and $\sum_{i=0}^{d-1}|\boldsymbol{\lambda}^{(i)}|=n$. We thus have $\boldsymbol{\lambda}^{(i)}=(\lambda^{(i,0)},\lambda^{(i,1)},\dots,\lambda^{(i,l-1)})$, where $\lambda^{(i,j)}$ is a partition for all $i=0,1,\dots,d-1$ and $j=0,1,\dots,l-1$, and $\sum_{i=0}^{d-1}\sum_{j=0}^{l-1}|\lambda^{(i,j)}|=n$.

A node $\boldsymbol{\theta}$ of $\boldsymbol{\lambda}$ is a 4-tuple (x,y,i,j), where $0\leqslant i\leqslant d-1$, $0\leqslant j\leqslant l-1$ and (x,y) is a node of the partition $\lambda^{(i,j)}$. We define $\mathrm{p}(\boldsymbol{\theta}):=i$ to be the *d-position* of $\boldsymbol{\theta}$ and $\mathrm{c}(\boldsymbol{\theta}):=Q_jq^{y-x}$ to be the *l-quantum content* of $\boldsymbol{\theta}$.

Following the definitions in §3.1, a *(d,l)-tableau* is simply a dl-tableau and a *standard (d,l)-tableau* is simply a standard dl-tableau. For a (d,l)-tableau $\mathcal{T}$ and for $i=1,\dots,n$, we denote respectively by $\mathrm{p}(\mathcal{T}|i)$ and $\mathrm{c}(\mathcal{T}|i)$ the d-position and the l-quantum content of the node with the number i in it.

Now, let $\boldsymbol{\lambda}\in\mathcal{P}(d,l,n)$, and let $V_{\boldsymbol{\lambda}}$ be a $\mathbb{C}(q)$-vector space with a basis $\{\mathbf{v}_{\mathcal{T}}\}$ indexed by the standard (d,l)-tableaux of shape $\boldsymbol{\lambda}$. We set $\mathbf{v}_{\mathcal{T}}:=0$ for any non-standard (d,l)-tableau $\mathcal{T}$ of shape $\boldsymbol{\lambda}$. By [ChPdA2, Propositions 3.2 & 3.4], the vector space $V_{\boldsymbol{\lambda}}$ is a representation of $\mathcal{K}_l\mathrm{Y}(d,l,n)$, with the action of the generators $g_1,\ldots,g_{n-1},t_1,\ldots,t_n$ on the basis element $\mathbf{v}_{\mathcal{T}}$ defined exactly as in Theorem 3.2, and the action of the generator X_1 given by:

$$X_1(\mathbf{v}_{\mathcal{T}}) = \mathrm{c}(\mathcal{T}|1)\,\mathbf{v}_{\mathcal{T}}. \tag{4.3}$$

Further, the set $\{V_{\boldsymbol{\lambda}} \,|\, \boldsymbol{\lambda} \in \mathcal{P}(d,l,n)\}$ is a complete set of pairwise non-isomorphic irreducible representations of $\mathcal{K}_l\mathrm{Y}(d,l,n)$.

Remark 4.5. We can easily show, by induction on i, that [ChPdA2, Lemma 3.3]:

$$X_i(\mathbf{v}_{\mathcal{T}}) = \mathrm{c}(\mathcal{T}|i)\,\mathbf{v}_{\mathcal{T}} \quad \text{for all } i=1,\ldots,n. \tag{4.4}$$

We also have a semisimplicity criterion for cyclotomic Yokonuma–Hecke algebras, which is exactly the same as Ariki's semisimplicity criterion [Ar] for Ariki–Koike algebras [ChPdA2, Proposition 4.7]:

Proposition 4.6. *Let $\theta : \mathcal{R}_l \to \mathbb{C}$ be a ring homomorphism such that $\theta(q)\prod_{j=0}^{l-1}\theta(Q_j) \neq 0$. The specialised cyclotomic Yokonuma–Hecke algebra $\mathbb{C}\mathrm{Y}(d,l,n)_\theta$, defined via θ, is (split) semisimple if and only if $\theta(P) \neq 0$, where*

$$P = \prod_{1\leqslant i\leqslant n}(1+q+\cdots+q^{i-1}) \prod_{0\leqslant s<t\leqslant l-1}\ \prod_{-n<k<n}(q^kQ_s - Q_t).$$

We deduce that the algebra $\mathbb{C}\mathrm{Y}(d,l,n)_\theta$ is semisimple if and only if the specialised Ariki–Koike algebra $\mathbb{C}\mathcal{H}(l,n)_\theta$ is semisimple.

Finally, we have proved the existence of a "canonical" symmetrising form on $\mathcal{K}_l\mathrm{Y}(d,l,n)$ and calculated the Schur elements with respect to it [ChPdA2, §7]:

Theorem 4.7. *We define the linear map $\tau : \mathrm{Y}(d,l,n) \to \mathcal{R}_l$ by*

$$\tau(X_1^{a_1}\ldots X_n^{a_n}t_1^{b_1}\ldots t_n^{b_n}g_w) = \begin{cases} 1 & \text{if } w=1 \text{ and } a_j=b_j=0 \text{ for all } j=1,2,\ldots,n, \\ 0 & \text{otherwise,} \end{cases} \tag{4.5}$$

where $w \in \mathfrak{S}_n$, $a_j \in Z_l$ and $0 \leqslant b_j \leqslant d-1$ for all $j=1,2,\ldots,n$. Then τ (extended linearly) is a symmetrising form on $\mathcal{K}_l\mathrm{Y}(d,l,n)$. If $\boldsymbol{\lambda} = (\boldsymbol{\lambda}^{(0)},\ldots,\boldsymbol{\lambda}^{(d-1)}) \in \mathcal{P}(d,l,n)$, then the Schur element of $V_{\boldsymbol{\lambda}}$ with respect to τ is

$$s_{\boldsymbol{\lambda}} = d^n\, s_{\boldsymbol{\lambda}^{(0)}}s_{\boldsymbol{\lambda}^{(1)}}\ldots s_{\boldsymbol{\lambda}^{(d-1)}}\,, \tag{4.6}$$

where $s_{\boldsymbol{\lambda}^{(i)}}$ is the Schur element of the Ariki–Koike algebra $\mathcal{H}(l,|\boldsymbol{\lambda}^{(i)}|)$ corresponding to $\boldsymbol{\lambda}^{(i)}$ for all $i = 0,1,\ldots,d-1$ (we take $s_{\boldsymbol{\emptyset}} := 1$).

The Schur elements of Ariki–Koike algebras have been calculated independently by Geck–Iancu–Malle [GIM] and Mathas [Mat]. A simple formula for them is given by Jacon and the author in [ChJa].

Remark 4.8. The map τ is known to be a symmetrising form on $\mathrm{Y}(d,l,n)$ (defined over $\mathcal{R}_l$) in cases $d=1$ [MalMat] and $l=1$ [ChPdA1]. In these cases, τ is called the *canonical symmetrising form* on $\mathrm{Y}(d,l,n)$.

Remark 4.9. Equation (4.6) hints towards an isomorphism between the cyclotomic Yokonuma–Hecke algebra $\mathrm{Y}(d,l,n)$ and a direct sum of matrix algebras over tensor products of Ariki–Koike algebras; this isomorphism was recently described by Poulain d'Andecy in [PdA] and by Rostam in [Ro].

5. Temperley–Lieb quotients of Yokonuma–Hecke algebras

The Temperley–Lieb algebra was introduced by Temperley and Lieb in [TeLi] for its applications in statistical mechanics. Jones [Jo1, Jo2, Jo3] later showed that it can be obtained as a quotient of the Iwahori–Hecke algebra $\mathcal{H}_n(q)$ of type A by a two-sided ideal, and used it for the construction of the knot invariant known as the Jones polynomial.

As explained in the introduction, we have three possible analogues of the Temperley–Lieb algebra in the Yokonuma–Hecke algebra setting: the Yokonuma–Temperley–Lieb algebra [GJKL1], the Framisation of the Temperley–Lieb algebra [GJKL2] and the Complex Reflection Temperley–Lieb algebra [GJKL2]. All three are defined as quotients of the Yokonuma–Hecke algebra $\mathrm{Y}_{d,n}(q)$ of type A by a suitable two-sided ideal, and they specialise to the classical Temperley–Lieb algebra for $d=1$.

In this section, we will determine the irreducible representations of the three algebras by showing which representations of $\mathrm{Y}_{d,n}(q)$ pass to each quotient. We will compute their dimensions and construct bases for them. At the end of this section, it will be clear that the most natural analogue of the Temperley–Lieb algebra in this setting is the Framisation of the Temperley–Lieb algebra.

First, let us recall some information about the classical setting. Let $n \geqslant 3$. The *Temperley–Lieb algebra* $\mathrm{TL}_n(q)$ is defined as the quotient of the Iwahori–Hecke algebra $\mathcal{H}_n(q) \cong \mathrm{Y}_{1,n}(q)$ by the ideal I_n generated by the elements

$$g_{i,i+1} := 1 + g_i + g_{i+1} + g_i g_{i+1} + g_{i+1} g_i + g_i g_{i+1} g_i = \sum_{w \in \langle s_i, s_{i+1} \rangle} g_w$$

for all $i = 1, \ldots, n-2$. It turns out that this ideal is principal, and we have $I_n = \langle\, g_{1,2}\, \rangle$.

Since the algebra $\mathbb{C}(q)\mathcal{H}_n(q)$ is semisimple, the algebra $\mathbb{C}(q)\mathrm{TL}_n(q)$ is also semisimple and its irreducible representations are precisely the irreducible representations of $\mathbb{C}(q)\mathcal{H}_n(q)$ that pass to the quotient. That is, for $\lambda \in \mathcal{P}(n)$, V_λ is an irreducible representation of $\mathbb{C}(q)\mathrm{TL}_n(q)$ if and only if $g_{1,2}(\mathrm{v}_{\mathcal{T}}) = 0$ for every standard tableau $\mathcal{T}$ of shape λ. It is easy to see that the latter is equivalent to the trivial representation not being a direct summand of the restriction $\mathrm{Res}^{\mathfrak{S}_n}_{\langle s_1, s_2 \rangle}(E^\lambda)$, where E^λ is the irreducible representation of the symmetric group $\mathfrak{S}_n$ labelled by λ. Since the restriction from $\mathfrak{S}_n$ to $\mathfrak{S}_3 \cong \langle s_1, s_2 \rangle$ corresponds to the simple removal of boxes from the Young diagram of λ, and the trivial representation of $\mathfrak{S}_3$ is labelled by the partition (3), we obtain the following description of the irreducible representations of $\mathbb{C}(q)\mathrm{TL}_n(q)$:

Theorem 5.1. *Let $\lambda \in \mathcal{P}(n)$. We have that V_λ is an irreducible representation of $\mathbb{C}(q)\mathrm{TL}_n(q)$ if and only if the Young diagram of λ has at most two columns.*

Now, let $n \in \mathbb{N}$, and let $\underline{i} = (i_1, \ldots, i_p)$ and $\underline{k} = (k_1, \ldots k_p)$ be two p-tuples of non-negative integers, with $0 \leqslant p \leqslant n-1$. We denote by $\mathfrak{H}_n$ the set of pairs $(\underline{i}, \underline{k})$ such that

$$1 \leqslant i_1 < i_2 < \cdots < i_p \leqslant n-1 \quad \text{and} \quad i_j - k_j \geqslant 1 \quad \forall\, j = 1, \ldots, p.$$

For $(\underline{i}, \underline{k}) \in \mathfrak{H}_n$, we set

$$g_{\underline{i},\underline{k}} := (g_{i_1} g_{i_1-1} \ldots g_{i_1-k_1})(g_{i_2} g_{i_2-1} \ldots g_{i_2-k_2}) \ldots (g_{i_p} g_{i_p-1} \ldots g_{i_p-k_p}) \in \mathcal{H}_n(q).$$

We take $g_{\emptyset,\emptyset}$ to be equal to 1. We have that the set

$$\mathcal{B}^{\mathrm{H}}_{1,n} = \{ g_w \,|\, w \in \mathfrak{S}_n \} = \{ g_{\underline{i},\underline{k}} \,|\, (\underline{i}, \underline{k}) \in \mathfrak{H}_n \}$$

is the standard basis of $\mathcal{H}_n(q)$ as a $\mathbb{C}[q, q^{-1}]$-module.

Further, let us denote by $\mathfrak{T}_n$ the subset of $\mathfrak{H}_n$ consisting of the pairs $(\underline{i}, \underline{k})$ such that

$$1 \leqslant i_1 < i_2 < \cdots < i_p \leqslant n-1 \quad \text{and} \quad 1 \leqslant i_1 - k_1 < i_2 - k_2 < \cdots < i_p - k_p \leqslant n-1.$$

Jones [Jo1] has shown that the set

$$\mathcal{B}^{\mathrm{TL}}_{1,n} := \{ g_{\underline{i},\underline{k}} \,|\, (\underline{i}, \underline{k}) \in \mathfrak{T}_n \}$$

is a basis of $\mathrm{TL}_n(q)$ as a $\mathbb{C}[q, q^{-1}]$-module. We have $|\mathcal{B}^{\mathrm{TL}}_n| = C_n$, where C_n is the n-th Catalan number, *i.e.*,

$$C_n = \frac{1}{n+1} \binom{2n}{n} = \frac{1}{n+1} \sum_{k=0}^{n} \binom{n}{k}^2.$$

5.1. **The Yokonuma–Temperley–Lieb algebra.** Let $d \in \mathbb{N}^*$ and let $n \in \mathbb{N}$ with $n \geqslant 3$. The *Yokonuma–Temperley–Lieb algebra* $\mathrm{YTL}_{d,n}(q)$ is defined as the quotient of the Yokonuma–Hecke algebra $\mathrm{Y}_{d,n}(q)$ by the ideal $I_{d,n} := \langle\, g_{1,2}\, \rangle$.

Since the algebra $\mathbb{C}(q)\mathrm{Y}_{d,n}(q)$ is semisimple, the algebra $\mathbb{C}(q)\mathrm{YTL}_{d,n}(q)$ is also semisimple and its irreducible representations are precisely the irreducible representations of $\mathbb{C}(q)\mathrm{Y}_{d,n}(q)$ that pass to the quotient. That is, for $\boldsymbol{\lambda} \in \mathcal{P}(d,n)$, $V_{\boldsymbol{\lambda}}$ is an irreducible representation of $\mathbb{C}(q)\mathrm{YTL}_{d,n}(q)$ if and only if $g_{1,2}(\mathbf{v}_{\mathcal{T}}) = 0$ for every standard d-tableau $\mathcal{T}$ of shape $\boldsymbol{\lambda}$. It is easy to see that the latter is equivalent to the trivial representation not being a direct summand of the restriction $\mathrm{Res}^{G(d,1,n)}_{\langle s_1, s_2 \rangle}(E^{\boldsymbol{\lambda}})$, where $E^{\boldsymbol{\lambda}}$ is the irreducible representation of the complex reflection group $G(d,1,n)$ labelled by $\boldsymbol{\lambda}$. Unfortunately, this restriction for $d > 1$ does not correspond to the simple removal of boxes from the Young diagram of $\boldsymbol{\lambda}$ (as in the symmetric group case), but it is controlled by the so-called *Littlewood–Richardson coefficients.* Using algebraic combinatorics, we obtain the following description of the irreducible representations of $\mathbb{C}(q)\mathrm{YTL}_{d,n}(q)$ [ChPo1, Theorem 3]:

Theorem 5.2. *Let $\boldsymbol{\lambda} = (\lambda^{(0)}, \dots, \lambda^{(d-1)}) \in \mathcal{P}(d,n)$. We have that $V_{\boldsymbol{\lambda}}$ is an irreducible representation of $\mathbb{C}(q)\mathrm{YTL}_{d,n}(q)$ if and only if the Young diagram of $\boldsymbol{\lambda}$ has at most two columns in total, that is, $\sum_{i=0}^{d-1} \lambda_1^{(i)} \leqslant 2$.*

Using the fact that the algebra $\mathbb{C}(q)\mathrm{YTL}_{d,n}(q)$ is semisimple and the above description of its irreducible representations, we have been able to calculate the dimension of the Yokonuma–Temperley–Lieb algebra [ChPo1, Proposition 4]. We have

$$\dim_{\mathbb{C}(q)}(\mathbb{C}(q)\mathrm{YTL}_{d,n}(q)) = \frac{n(d^2-d)+d^2+d}{2}\, C_n - (d^2-d).$$

What is more, we have shown in [ChPo1] that $\mathrm{YTL}_{d,n}(q)$ is a free $\mathbb{C}[q,q^{-1}]$-module of rank equal to the dimension above. However, note that, even though the set

$$\mathcal{B}^{\mathrm{H}}_{d,n} = \left\{\, t_1^{r_1} \dots t_n^{r_n} g_{\underline{i},\underline{k}} \mid (\underline{i},\underline{k}) \in \mathfrak{H}_n,\ 0 \leqslant r_j \leqslant d-1 \text{ for all } j = 1,2,\dots,n \right\}$$

is a basis of $\mathrm{Y}_{d,n}(q)$ as a $\mathbb{C}[q,q^{-1}]$-module, the set

$$\mathcal{B}^{\mathrm{TL}}_{d,n} = \left\{\, t_1^{r_1} \dots t_n^{r_n} g_{\underline{i},\underline{k}} \mid (\underline{i},\underline{k}) \in \mathfrak{T}_n,\ 0 \leqslant r_j \leqslant d-1 \text{ for all } j = 1,2,\dots,n \right\}$$

is not a basis of $\mathrm{YTL}_{d,n}(q)$ as a $\mathbb{C}[q,q^{-1}]$-module, since $|\mathcal{B}^{\mathrm{TL}}_{d,n}| = d^n C_n$. The set $\mathcal{B}^{\mathrm{TL}}_{d,n}$ is simply a generating set for $\mathrm{YTL}_{d,n}(q)$, and we have managed to find a subset $\mathcal{B}^{\mathrm{YTL}}_{d,n}$ of $\mathcal{B}^{\mathrm{TL}}_{d,n}$ that is a basis of $\mathrm{YTL}_{d,n}(q)$ by proving the following remarkable property: Let $(\underline{i},\underline{k}) \in \mathfrak{T}_n$. We denote by $\mathcal{I}(g_{\underline{i},\underline{k}})$ the set (without repetition) of all indices of the g_j's appearing in $g_{\underline{i},\underline{k}}$, *i.e.*,

$$\mathcal{I}(g_{\underline{i},\underline{k}}) = \{i_1, i_1-1, \dots, i_1-k_1, i_2, i_2-1, \dots, i_2-k_2, \dots, i_p, i_p-1, \dots, i_p-k_p\}.$$

We define the *weight* of $g_{\underline{i},\underline{k}}$ to be $w_{\underline{i},\underline{k}} := |\mathcal{I}(g_{\underline{i},\underline{k}})|$. We then have [ChPo1, Propositions 9, 11, 12]:

$$|\{(r_1,\dots,r_n) \in \{0,\dots,d-1\}^n \,|\, t_1^{r_1} \dots t_n^{r_n} g_{\underline{i},\underline{k}} \in \mathcal{B}^{\mathrm{YTL}}_{d,n}\}| = 2^{n-w_{\underline{i},\underline{k}}-1}(d^2-d) + d - \delta_{w_{\underline{i},\underline{k}},0}(d^2-d),$$

where $\delta_{i,j}$ stands for Kronecker's delta (note that we have $w_{\underline{i},\underline{k}} = 0$ if and only if $g_{\underline{i},\underline{k}} = 1$). Thanks to this property, an explicit basis for $\mathrm{YTL}_{d,n}(q)$ as a $\mathbb{C}[q,q^{-1}]$-module is described in [ChPo1].

5.2. **The Framisation of the Temperley–Lieb algebra.** Let $d \in \mathbb{N}^*$ and let $n \in \mathbb{N}$ with $n \geqslant 3$. The *Framisation of the Temperley–Lieb algebra* $\mathrm{FTL}_{d,n}(q)$ is defined as the quotient of the Yokonuma–Hecke algebra $\mathrm{Y}_{d,n}(q)$ by the ideal $J_{d,n} := \langle\, e_1 e_2 g_{1,2} \,\rangle$. We remark that $J_{d,n}$ can be also defined as the ideal generated by the element $\sum_{0\leqslant a,b\leqslant d-1} t_1^a t_2^b t_3^{-a-b}\, g_{1,2}$.

Again, since the algebra $\mathbb{C}(q)\mathrm{Y}_{d,n}(q)$ is semisimple, the algebra $\mathbb{C}(q)\mathrm{FTL}_{d,n}(q)$ is also semisimple and its irreducible representations are precisely the irreducible representations of $\mathbb{C}(q)\mathrm{Y}_{d,n}(q)$ that pass to the quotient. That is, for $\boldsymbol{\lambda} \in \mathcal{P}(d,n)$, $V_{\boldsymbol{\lambda}}$ is an irreducible representation of $\mathbb{C}(q)\mathrm{FTL}_{d,n}(q)$ if and only if $e_1 e_2 g_{1,2}(\mathbf{v}_{\mathcal{T}}) = 0$ for every standard d-tableau $\mathcal{T}$ of shape $\boldsymbol{\lambda}$. Using the formulas for the irreducible representations of $\mathbb{C}(q)\mathrm{Y}_{d,n}(q)$ given by Theorem 3.2, we obtain the following description of the irreducible representations of $\mathbb{C}(q)\mathrm{FTL}_{d,n}(q)$ [ChPo2, Theorem 3.10]:

Theorem 5.3. *Let $\boldsymbol{\lambda} = (\lambda^{(0)}, \dots, \lambda^{(d-1)}) \in \mathcal{P}(d,n)$. We have that $V_{\boldsymbol{\lambda}}$ is an irreducible representation of $\mathbb{C}(q)\mathrm{FTL}_{d,n}(q)$ if and only if the Young diagram of $\lambda^{(i)}$ has at most two columns for all $i = 0, \dots, d-1$.*

Following the recipe of [JaPdA, §3], we have proved the following isomorphism theorem for $\mathrm{FTL}_{d,n}(q)$ [ChPo2, Theorem 4.3]:

Theorem 5.4. *There exists a $\mathbb{C}[q,q^{-1}]$ algebra isomorphism*

$$\psi_n : \mathrm{FTL}_{d,n}(q) \to \bigoplus_{\mu \in \mathrm{Comp}_d(n)} \mathrm{Mat}_{m_\mu}(\mathrm{TL}_{\mu_0}(q) \otimes \mathrm{TL}_{\mu_1}(q) \otimes \cdots \otimes \mathrm{TL}_{\mu_{d-1}}(q)),$$

where $\mathrm{Comp}_d(n)$ and m_μ are as defined in (3.15) and (3.16), and we take $\mathrm{TL}_n(q) \cong \mathcal{H}_n(q)$ for $n < 3$.

We deduce that the following set is a basis of $\mathrm{FTL}_{d,n}(q)$ as a $\mathbb{C}[q,q^{-1}]$-module [ChPo2, Proposition 4.4]:

$$\left\{\psi_n^{-1}(b_0^\mu b_1^\mu \dots b_{d-1}^\mu\, M_{k,l}^\mu)\,|\,\mu\in \mathrm{Comp}_d(n), b_i^\mu \in \mathcal{B}^{\mathrm{TL}}_{1,\mu_i} \text{ for all } i=0,\dots,d-1, 1\leqslant k,l\leqslant m_\mu\right\},$$

where $M_{k,l}^\mu$ denotes the elementary $m_\mu\times m_\mu$ matrix with 1 in position (k,l). In particular, $\mathrm{FTL}_{d,n}(q)$ is a free $\mathbb{C}[q,q^{-1}]$-module of rank

$$\sum_{\mu\in\mathrm{Comp}_d(n)} m_\mu^2\, C_{\mu_0}C_{\mu_1}\cdots C_{\mu_{d-1}}.$$

5.3. **The Complex Reflection Temperley–Lieb algebra.** Let $d\in\mathbb{N}^*$ and let $n\in\mathbb{N}$ with $n\geqslant 3$. The *Complex Reflection Temperley–Lieb algebra* $\mathrm{CTL}_{d,n}(q)$ is defined as the quotient of the Yokonuma–Hecke algebra $\mathrm{Y}_{d,n}(q)$ by the ideal $K_{d,n}:=\langle\,\sum_{s=0}^{d-1} t_1^s e_1e_2g_{1,2}\,\rangle$. We remark that $K_{d,n}$ can be also viewed as the ideal generated by the element $\sum_{0\leqslant a,b,c\leqslant d-1} t_1^a t_2^b t_3^c\, g_{1,2}$.

Once more, the algebra $\mathbb{C}(q)\mathrm{CTL}_{d,n}(q)$ is semisimple and, for $\boldsymbol{\lambda}\in\mathcal{P}(d,n)$, $V_{\boldsymbol{\lambda}}$ is an irreducible representation of $\mathbb{C}(q)\mathrm{CTL}_{d,n}(q)$ if and only if $\sum_{s=0}^{d-1} t_1^s e_1e_2g_{1,2}(\mathbf{v}_{\mathcal{T}})=0$ for every standard d-tableau $\mathcal{T}$ of shape $\boldsymbol{\lambda}$. Using the formulas for the irreducible representations of $\mathbb{C}(q)\mathrm{Y}_{d,n}(q)$ given by Theorem 3.2, we obtain the following description of the irreducible representations of $\mathbb{C}(q)\mathrm{CTL}_{d,n}(q)$ [ChPo2, Theorem 5.3]:

Theorem 5.5. *Let $\{\xi_0,\xi_1,\dots,\xi_{d-1}\}$ be the set of all d-th roots of unity (ordered arbitrarily) as in Theorem 3.2. Let $i_0\in\{0,\dots,d-1\}$ be such that $\xi_{i_0}=1$, and let $\boldsymbol{\lambda}=(\lambda^{(0)},\dots,\lambda^{(d-1)})\in\mathcal{P}(d,n)$. We have that $V_{\boldsymbol{\lambda}}$ is an irreducible representation of $\mathbb{C}(q)\mathrm{CTL}_{d,n}(q)$ if and only if the Young diagram of $\lambda^{(i_0)}$ has at most two columns.*

Following the recipe of [JaPdA, §3], we have proved the following isomorphism theorem for $\mathrm{CTL}_{d,n}(q)$ [ChPo2, Theorem 5.8]:

Theorem 5.6. *There exists a $\mathbb{C}[q,q^{-1}]$ algebra isomorphism*

$$\overline{\psi}_n:\mathrm{CTL}_{d,n}(q)\to\bigoplus_{\mu\in\mathrm{Comp}_d(n)}\mathrm{Mat}_{m_\mu}(\mathrm{TL}_{\mu_0}(q)\otimes\mathcal{H}_{\mu_1}(q)\otimes\mathcal{H}_{\mu_2}(q)\otimes\cdots\otimes\mathcal{H}_{\mu_{d-1}}(q)),$$

where $\mathrm{Comp}_d(n)$ *and* m_μ *are as defined in* (3.15) *and* (3.16), *and we take* $\mathrm{TL}_n(q)\cong\mathcal{H}_n(q)$ *for* $n<3$.

We deduce that the following set is a basis of $\mathrm{CTL}_{d,n}(q)$ as a $\mathbb{C}[q,q^{-1}]$-module [ChPo2, Proposition 5.9]:

$$\left\{\overline{\psi}_n^{-1}(b_0^\mu b_1^\mu \dots b_{d-1}^\mu\, M_{k,l}^\mu)\,|\,\mu\in \mathrm{Comp}_d(n), b_0^\mu \in \mathcal{B}^{\mathrm{TL}}_{1,\mu_0}, b_i^\mu\in\mathcal{B}^{\mathrm{H}}_{1,\mu_i} \text{ for all } i=1,\dots,d-1, 1\leqslant k,l\leqslant m_\mu\right\},$$

where $M_{k,l}^\mu$ denotes the elementary $m_\mu\times m_\mu$ matrix with 1 in position (k,l). In particular, $\mathrm{CTL}_{d,n}(q)$ is a free $\mathbb{C}[q,q^{-1}]$-module of rank

$$\sum_{\mu\in\mathrm{Comp}_d(n)} m_\mu^2\, C_{\mu_0}\mu_1!\dots\mu_{d-1}!.$$

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